Monkey Doing A Multiple Choice Exam

22 December 2023

Problem statement

Suppose, you were given a multiple choice exam, but you didn't feel like doing it, so you just selected random answers. What grade will you get? 25%? 0%? 100%?

We must realise that you cannot predict exactly what score you will get, however, you can find the probability of getting a certain number of questions correct. A complete statement of the problem is as follows:

If you were to randomly select the options on a multiple choice exam consisting of n questions and m otpions per question with only one correct option, what is the probability of you getting k out of n questions correct?

The extreme cases

It is easiest to find the probability of getting all questions correct or the probability of getting no questions correct.

Say, you randomly choose one option out of 4 possible options on a single question. Assume there is only one correct option. The probability of you guessing correctly is $\frac{1}{4}$. Now, you move on to the next question. The probability of you guessing the first question correctly **and** the second one as well is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. The probability of guessing three questions correctly is $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$. If you were to answer a test with 20 questions in total, then the probability of getting a perfect score would be $\left(\frac{1}{4}\right)^{20}$.

In general, the probability ($P_{perfect}$) of getting a perfect score by selecting options randomly on a multiple choice exam with n questions and m options per question with only one correct option is given by

$$P_{perfect} = \left(\frac{1}{m}\right)^n.$$

Similar reasoning will allow us to find the probability of getting a score of 0. The probability of getting a single question wrong with m options per question and only once correct option is $\frac{m-1}{m}$. Therefore:

In general, the probability (P_{none}) of getting no questions correct by selecting options randomly on a multiple choice exam with n questions and m options per question with only one correct option is given by

$$\begin{aligned} P_{none} &= \left(\frac{m-1}{m}\right)^n \\ &= \left(1 - \frac{1}{m}\right)^n \\ &= (1 - P_c)^n \end{aligned}$$

where P_c is the probability of answering a single question correctly.

The complete solution

Lets now consider cases such as getting 5 out of 20 questions correct.

Assume there are still 4 options per question. You could guess the first 5 questions correctly while getting the other 15 wrong. The probability of doing so is $\left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{15}$. However, this isn't the only way to get a score of 5 out of 20. You could also guess the first 3 correct, the next 15 wrong and the last 2 correct. The same score could be achieved with many different arrangements of wrong and correct guesses. The probability for each one of these arrangements is the same though. To figure out a complete solution to our problem, we need to find the number of possible arrangements of wrong and correct guesses.

Let C represent a correct guess and W represent a wrong guess. A possible arrangement of wrong and correct guesses may be represented as a list:

$$(C,\ C,\ \ldots,\ C,\ W,\ W,\ \ldots,\ W)$$

where a C in position 1 indicates that question 1 was guessed correctly.

To find the number of arrangments is to find the number of permutations. In our case, not every object is unique. The number of permutations of n objects where p are of one kind, q are of another kind, r are of yet another kind, and so on, is given by

$$\frac{n!}{p! \cdot q! \cdot r! \cdot \ldots}.$$

With our situation of 5 correct guesses out of 20, the number of arrangements of such guesses is $\frac{20!}{5! \cdot 15!}$.

The probability of each of the arrangements was found earlier. The probability of getting a score of 5 out of 20 can be found by adding the probabilities for all the different arrangements.

$$\begin{aligned} P(\textit{guessing } 5 \textit{ out of } 20 \textit{ correct}) &= P(\textit{arrangement}_1 \textbf{ or } \textit{arrangement}_2 \textbf{ or } \ldots) \\ &= P(\textit{arrangement}_1) + P(\textit{arrangement}_2) + \ldots \\ &= \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{15} \left(\frac{20!}{5! \cdot 15!}\right). \end{aligned}$$

All of our reasoning above can be generalised.

With m options per question and only one correct option, the probability of guessing k out of n questions correctly in a specific way is

$$\left(\frac{1}{m}\right)^k \left(\frac{m-1}{m}\right)^{n-k}.$$

The number of ways to do so is

$$\frac{n!}{k!(n-k)!}.$$

Therefore,

The probability, $f(k,\ n,\ m)$, of getting k out of n questions correct by selecting options randomly on a multiple choice exam where there are m options per question and only one correct option is

$$f(k,\ n,\ m) = \left(\frac{1}{m}\right)^{k} \left(\frac{m-1}{m}\right)^{n-k} \left(\frac{n!}{k!(n-k)!}\right) \quad \diamond$$

This is the answer to the problem stated at the beggining of this article.

Notice that setting $k = n$ in the above expression yields the expression for $P_{perfect}$ and setting $k = 0$ yields the expression for P_{none}.