On the dimensional consistency of physics textbook equations

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Introduction

In this paper, I will talk about the apparent dimensional inconsistency of some equations which can be found in physics textbooks. I will show how certain formulations make no sense when subjected to dimensional analysis and how this can be explained via so called "hidden units".

The problem

When reading physics textbooks, it is common to see questions such as the following

Let $r(t) = 6t^2\ m$ represent the position of a particle moving along a straight line. Calculate the position and velocity of the particle at $t = 5\ s$.

It is not difficult to observe some problems with these types of mathematical formulations. When trying to solve for time at $t = 5\ s$ and plugging in the $5\ s$ directly into the function, we get the following

$$\begin{aligned} r(5\ s) &= 6(5\ s)^2\ m \\ &= 6(25\ s^2)\ m \\ &= 150\ s^2 m \end{aligned}$$

The units in the answer here make no sense. Position should be given in metres.

In most physics courses, we are taught to carry the units throughout a calculation if we want to find the correct units for the answer. Units behave like constants and therefore can be manipulated algebraically. Do the rules for unit manipulation break down in problems such as the ones shown here?

Sometimes textbooks present a position function without any units associated with it such as $r(t) = 6t^2$. This equation will also become problematic once you plugin $t = 5\ s$.

The only somewhat non-problematic formulation is one where the function is given as $r(t) = 6t^2\ m$ and t is treated as a dimensionless quantity:

$$\begin{aligned} r(5) &= 6(5)^2\ m \\ &= 150\ m \end{aligned}$$

While this answer makes sense, it seems like a "hack" to think of t as dimensionless.

Adding to the confusion, most of the equations presented here are dimensionally inconsistent:

$$\begin{aligned} r(t) &= 6t^2\ m \\ [r(t)] &= [6t^2\ m] \\ L &= T^2 L \quad (inconsistent) \end{aligned}$$

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$$\begin{aligned} r(t) &= 6t^2\ m \quad (t\ dimensionless) \\ [r(t)] &= [6t^2\ m] \\ L &= L \quad (consistent,\ but\ seems\ like\ a\ "hack") \end{aligned}$$

The solution

It looks like all the equations presented here make no sense. The explanation to this inconsistency is that there exist "hidden" or "implied" units within the equation.

When we say $r(t) = 6t^2$ or $r(t) = 6t^2\ m$

what we actually mean can be one of the following

$r(t) = 6\ m/s^2\ t^2$ or $r(t) = \frac{6}{s^2}\ t^2\ m$ or $r(t) = 6(\frac{t}{1\ s})^2\ m$

All three of these formulations are equivalent.

$$\begin{aligned} 6\ m/s^2\ t^2 &= \frac{6}{s^2}\ t^2\ m \\ &= 6\left(\frac{t^2}{s^2}\right)\ m \\ &= 6\left(\frac{t}{1\ s}\right)^2\ m \end{aligned}$$

These equations that include the hidden units are dimensionally consistent:

$$\begin{aligned} r(t) &= 6\ m/s^2\ t^2 \\ [r(t)] &= [6\ m/s^2\ t^2] \\ L &= LT^{-2}T^2 \\ L &= L \end{aligned}$$

They also make sense when you plugin values of t:

$$\begin{aligned} r(5\ s) &= 6\ m/s^2\ (5\ s)^2 \\ &= 6\ m/s^2\ 25\ s^2 \\ &= 150\ m \end{aligned}$$

Moreover, applying calculus to such equations also makes sense:

$$\begin{aligned} v(t) &= \frac{d}{dt}r(t) \\ &= \frac{d}{dt}(6\ m/s^2\ t^2) \\ &= 6\ m/s^2\ \frac{d}{dt}t^2 \\ &= 12\ m/s^2\ t \end{aligned}$$

At first, there seems to be something wrong with the solution. How can velocity be presented with units of acceleration? Firstly, note that the equation is dimensionally consistent:

$$\begin{aligned} v(t) &= 12\ m/s^2\ t \\ [v(t)] &= [12\ m/s^2\ t] \\ LT^{-1} &= LT^{-2}T \\ LT^{-1} &= LT^{-1} \end{aligned}$$

It therefore makes sense when you plugin values of t:

$$\begin{aligned} v(1\ s) &= 12\ m/s^2\ (1\ s) \\ &= 12\ m/s \end{aligned}$$

The explanation for the appearance of m/s^2 in an expression for velocity is that it is a consequence of us explicitly specifying the hidden units in the expression for position. Normally, we would write such an equation for velocity as $v(t) = 12t\ m/s$ or even $v(t) = 12t$ for simplicity. Some of the units become hidden.

Another way to see the hidden units is to realise that the position function $r(t) = 6t^2$ that we have dealt with is in the form

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 \tag{1}$$

Which is the equation for position with constant acceleration. In the case of $r(t)$, x_0 and v_0 are set to zero and $\frac{1}{2} a_0 t^2 = 6\ m/s^2\ t^2$.

Differentiating the function $x(t)$ we get

$$\begin{aligned} v(t) &= \frac{d}{dt} x(t) \\ &= v_0 + a_0 t \end{aligned}$$

From this, we can see that the 12 that we have obtained by differentiating $r(t)$ was in fact a magnitude of acceleration, hence why we had the units m/s^2 associated with it.

conclusion

I have shown via examples that adding hidden units makes equations dimensionally consistent. I have only dealt with a single function, but I am sure that any function can be made consistent. If someday I find a function which cannot be made consistent or which somehow doesn't make sense even with the addition of hidden units, then maybe I'll have to revise my approaches taken in this paper.

Concerning functions where t appears in powers higher than 2, we will have to introduce units such as m/s^3. This is the unit of a physical quantity called "jerk", which is the derivative of acceleration. The equation for position with constant jerk is similar to equation (1) for position with constant acceleration, except that it has more terms.

If equations involving hidden units are so inconsistent, then why do we use them? The answer is simplicity. Putting a lot of units into expressions and then trying to intergrate or differentiate them will produce a lot of paper work, which could be avoided by writing quantities without units. Given $r(t) = 6t^2$, we can find $v(t) = 12t$ and then compute $r(5) = 150\ m$ and $v(5) = 60\ m/s$. This is the answer to the question stated at the beggining of this paper.